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A NOTE ON ADJOINT SEMIGROUPS ASSOCIATED WITH SOME LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Set $E = \mathbb{R}^n$ or \mathbb{C}^n ,

$$C_\gamma = \{\varphi: (-\infty, 0] \rightarrow E : \varphi \text{ is continuous and } a_\gamma(\varphi) = \lim_{\theta \rightarrow -\infty} e^{\gamma\theta} \varphi(\theta) \text{ exists}\},$$

where $\gamma \in \mathbb{R}$, and set

$$|\varphi|_\gamma = \sup\{e^{\gamma\theta} |\varphi(\theta)| : -\infty < \theta \leq 0\} \quad \varphi \in C_\gamma.$$

Then C_γ is a Banach space with respect to the norm $|\cdot|_\gamma$, and it is isomorphic to the Banach space $C = C([-1, 0], E)$: an isometric isomorphism ι_γ from C_γ onto C is given by

$$(\iota_\gamma \varphi)(s) = \begin{cases} e^{\gamma s/1+s} \varphi(s/1+s) & -1 < s \leq 0 \\ a_\gamma(\varphi) & s = -1. \end{cases}$$

The t -segment x_t of a function $x: (-\infty, t] \rightarrow E$ is a function, on $(-\infty, 0]$ into E , defined by $x_t(\theta) = x(t+\theta)$ $\theta \leq 0$. Consider a linear functional differential equation with the phase space C_γ :

$$(1) \quad \dot{x} = L(x_t),$$

where L is a continuous linear operator on C_γ into E . It is well known [3,4] that this equation has a unique solution $x(\varphi)$ satisfying the initial condition $x_0 = \varphi$ for any φ in C_γ , and that the one parameter family of operators $T(t)$, $t \geq 0$, defined by

$$T(t)\varphi = x_t(\varphi) \quad \text{for } \varphi \in C_\gamma$$

is a strongly continuous semigroup of bounded linear operators on C_γ . This semigroup is called the solution semigroup of Equation (1). We consider the representation of the adjoint semigroup of $T(t)$.

For linear functional differential equations with finite delays on the phase space C , it is shown [2] that C is sun-reflexive with respect to the solution semigroups. On the other hand, for the case $\gamma = 0$, the space C_0 is not sun-reflexive with respect to $T(t)$, [5]. The space C_0^{∞} defined below is isomorphic to the space $E \times BU$, where BU is the space of bounded, uniformly continuous functions on $(-\infty, 0]$ into E with the supremum norm; C_0 is imbedded onto the subspace $\{(a_0(\varphi), \varphi) : \varphi \in C_0\} \subset E \times BU$. Diekmann and Greiner suggested that, for any γ , the space C_γ would not be sun-reflexive with respect to $T(t)$ from the information about the spectrum of the resolvent $(\lambda I - A)^{-1}$, where A is the

infinitesimal generator of $T(t)$. What is $C_\gamma^{\circ\circ}$? This note is an unaccomplished approach to this question.

Before proceeding, we give the definition of sun-reflexivity. For a moment, denote by $T(t)$ a strongly continuous semigroup of bounded linear operators on a Banach space X . Then the adjoints semigroup $T^*(t)$ is not necessarily a strongly continuous on X^* . The maximal closed subspace of X^* on which $T^*(t)$ becomes a strongly continuous semigroup is the one defined by

$$X^\circ = \{x^\circ \in X^*: \lim_{t \rightarrow 0+} |T^*(t)x^\circ - x^\circ| = 0\}.$$

It is known that this space coincides with the closure of the $\mathcal{D}(A^*)$, the domain of the adjoint operator of the infinitesimal generator A of $T(t)$. Denote by $T^\circ(t)$ the restriction of $T^*(t)$ on X° . Repeating the above process for $T^\circ(t)$, we have a strongly continuous semigroup $T^{\circ\circ}(t)$ which is the restriction of $T^{\circ*}(t)$ on the space $X^{\circ\circ} = \{x^{\circ\circ} \in X^{\circ*}: \lim_{t \rightarrow 0+} |T^{\circ*}(t)x^{\circ\circ} - x^{\circ\circ}| = 0\}$. The original space X is isomorphically imbedded to a subspace of $X^{\circ\circ}$, and $T^{\circ\circ}(t)$ is an extension of $T(t)$. If X is isomorphic to the whole space $X^{\circ\circ}$, we say that X is sun-reflexive with respect to the semigroup $T(t)$. This occurs if and only if the resolvent $(\lambda I - A)^{-1}$ is weakly compact for any λ in the resolvent set $\rho(A)$, see [6]. Refer the book [1] for the theory of adjoint semigroups. The space $X^{\circ\circ}$ depends on $T(t)$; we may obtain another space as $X^{\circ\circ}$ from a different

semigroup. But for the solution semigroup of Equation (1), the spaces C_γ^\ominus , $C_\gamma^{\ominus\ominus}$ are independent of the choice of the linear operator L . This fact holds for solution semigroups of linear functional differential equations with infinite delays on the very general phase space, [5].

Now we return to the solution semigroup $T(t)$ of Equation (1). It is easy to show the following result, cf [5].

Theorem 1. A function φ in C_γ is in $\mathcal{D}(A)$ if and only if it is continuously differentiable, $\dot{\varphi}$ is in C_γ and $\dot{\varphi}(0) = L(\varphi)$; and $A\varphi = \dot{\varphi}$ for φ in $\mathcal{D}(A)$.

To obtain the information about the space $C_\gamma^{\ominus\ominus}$, we take $L = 0$ in Equation (1); that is, we consider the equation

$$(2) \quad \dot{x} = 0.$$

Denote by $S(t)$ the solution semigroup of this equation, and by B its infinitesimal generator. As claimed in the above, $\mathcal{D}(A)$ and $\mathcal{D}(B)$ have the same closure in C_γ^* , which we denote by C_γ^\ominus . Since the case that $\gamma = 0$ is already treated in [5], we assume that $\gamma \neq 0$ hereafter.

If φ is in C_γ , the function u defined by $u(\theta) = e^{-\gamma\theta}\varphi(\theta)$ for θ in $(-\infty, 0]$ is in C_0 . Set $j_\gamma(\varphi) = u$. Then j_γ is an isometric isomorphism between C_γ and C_0 . We characterize $\mathcal{D}(B)$ by the condition on u .

Theorem 2. Assume that $\gamma \neq 0$. Then φ in C_γ is in $\mathcal{D}(B)$ if and only if $u = j_\gamma(\varphi)$ is represented as

$$(3) \quad u(\theta) = \frac{1}{\gamma} v(0) - \int_{\theta}^0 v(s) ds \quad \text{for } \theta \text{ in } (-\infty, 0]$$

and for some function v in C_0 such that

$$(4) \quad \lim_{\theta \rightarrow -\infty} v(\theta) = 0 \quad \text{and} \quad \lim_{r \rightarrow -\infty} \int_r^0 v(s) ds \text{ converges.}$$

Proof. A function φ in C_γ is continuously differentiable if and only if u is so, and $\dot{\varphi}(\theta) = e^{-\gamma\theta} \{-\gamma u(\theta) + \dot{u}(\theta)\}$. Thus $a_\gamma(\dot{\varphi})$ exists if and only if $\lim_{\theta \rightarrow -\infty} \{-\gamma u(\theta) + \dot{u}(\theta)\}$ exists; and $\dot{\varphi}(0) = 0$ if and only if $-\gamma u(0) + \dot{u}(0) = 0$. Set $v(\theta) = \dot{u}(\theta)$. Then the last condition is equivalent that u is represented as in the above form (3). The condition that u is in C_0 is equivalent that v has the second condition in (4). If $u(\theta)$ and $-\gamma u(\theta) + \dot{u}(\theta)$ converge as $\theta \rightarrow -\infty$, then $u(\theta) \rightarrow 0$ as $\theta \rightarrow -\infty$. Thus we have the first condition for v in (4).

If η is a function of bounded variation on $[-1, 0]$ and φ is continuous on $[-1, 0]$, we can write

$$\int_{-1}^0 d\eta(t)\varphi(t) = [\eta(-1+) - \eta(-1)]\varphi(-1) + \lim_{r \rightarrow -1+} \int_r^0 d\eta(t)\varphi(t),$$

where $\eta(-1+) = \lim_{t \rightarrow -1+} \eta(t)$. Since $i_0 : C_0 \rightarrow C$ is an

isomorphism, $\iota_0^* : C^* \rightarrow C_0^*$ is also an isomorphism. Hence, from Riesz's representation theorem, for any u^* in C_0^* there exist a vector a in E^* , and a E^* -valued function f , of bounded variation on $(-\infty, 0]$, such that

$$\langle u^*, u \rangle = au(-\infty) + \int_{-\infty}^0 df(\theta)u(\theta) \quad \text{for } u \text{ in } C_0,$$

where $u(-\infty) = \lim_{\theta \rightarrow -\infty} u(\theta)$ and $\int_{-\infty}^0 = \lim_{r \rightarrow -\infty} \int_r^0$. Of course, the norm of u^* is given by

$$\|u^*\| = \|a\| + \text{Var}(f, (-\infty, 0]) = \|a\| + \lim_{r \rightarrow -\infty} \text{Var}(f, [r, 0]).$$

If f is normalized in the sense that $f(0) = 0$ and f is left continuous on $(-\infty, 0)$, the pair (a, f) is uniquely determined by u^* . Denote by NBV the class of those normalized functions. We may identify the space C_0^* with $E^* \times \text{NBV}$, and regard the isomorphism j_γ^* as the one from $E^* \times \text{NBV}$ to C_γ^* : $\langle j_\gamma^*(a, f), \varphi \rangle = \langle (a, f), u \rangle$, where $u = j_\gamma(\varphi)$. Namely the space $E^* \times \text{NBV}$ is the coordinate space of C_γ^* .

Theorem 3. Assume that $\gamma \neq 0$. Then an element (a, f) in $E^* \times \text{NBV}$ is a coordinate of an element of $\mathcal{Q}(B)$ if and only if a is arbitrary, $f(\theta)$ is absolutely continuous on $(-\infty, 0)$, and the equivalence class of $\dot{f}(\theta)$ in $L^1((-\infty, 0), E^*)$ contains a function which is in NBV and converges to 0 as $\theta \rightarrow -\infty$.

For such a (a, f) the coordinate (b, g) of $B^*(j_\gamma^*(a, f))$ is given by

$$(5) \quad b = -\gamma a$$

$$(6) \quad g(\theta) = \begin{cases} 0 & \text{for } \theta = 0 \\ -\gamma[f(\theta) - f(0-)] - \dot{f}(\theta) & \text{for } \theta < 0, \end{cases}$$

where we read that \dot{f} stands for the function in NBV mentioned in the above.

Proof. Consider the condition

$$(7) \quad j_\gamma^*(a, f) \text{ is in } \mathfrak{D}(B^*) \text{ and } B^*(j_\gamma^*(a, f)) = j_\gamma^*(b, g),$$

for (a, f) and (b, g) in $E^* \times \text{NBV}$. By the definition of B^* , this means that, for every φ in $\mathfrak{D}(B)$, $\langle j_\gamma^*(a, f), B\varphi \rangle = \langle j_\gamma^*(b, g), \varphi \rangle$, or $\langle (a, f), -\gamma u + \dot{u} \rangle = \langle (b, g), u \rangle$, where $u = j_\gamma(\varphi)$. The condition $f, g \in \text{NBV}$ implies that $f(0) = g(0) = 0$; and the condition $\varphi \in \mathfrak{D}(B)$ implies that $\dot{u}(-\infty) = 0$. From integration by parts we then have that

$$\langle (a, f), -\gamma u + \dot{u} \rangle = -[a - f(-\infty)]\gamma u(-\infty) + \int_{-\infty}^0 [\gamma f(\theta) d\theta + df(\theta)] \dot{u}(\theta)$$

$$\langle (b, g), u \rangle = [b - g(-\infty)]u(-\infty) - \int_{-\infty}^0 dg(\theta) \dot{u}(\theta).$$

Since u is represented as in (3), it holds that

$$u(-\infty) = \frac{1}{\gamma}v(0) - \int_{-\infty}^0 v(\theta)d\theta.$$

Hence we can write

$$\int_{-\infty}^0 [df(\theta) + \gamma f(\theta)d\theta]v(\theta) = -c\{\frac{1}{\gamma}v(0) - \int_{-\infty}^0 v(\theta)d\theta\} - \int_{-\infty}^0 dg(\theta)v(\theta),$$

where

$$c = \gamma(f(-\infty) - a) + g(-\infty) - b.$$

Furthermore, if we define a function h in NBV by

$$h(\theta) = \begin{cases} 0 & \theta = 0 \\ \frac{c}{\gamma} & \theta < 0, \end{cases}$$

we have that

$$(8) \quad \int_{-\infty}^0 [df(\theta) + \gamma f(\theta)d\theta]v(\theta) = \int_{-\infty}^0 [dh(\theta) + cd\theta - dg(\theta)]v(\theta).$$

Consequently Condition (7) is equivalent that Relation (8) holds for every v in C_0 having Property (4). Since every function with compact support has this property for v , it follows that $df(\theta) + \gamma f(\theta) = dh(\theta) + cd\theta - dg(\theta)$, or

$$(9) \quad f(\theta) - \gamma \int_{\theta}^0 f(s) ds = h(\theta) + c\theta + \int_{\theta}^0 g(s) ds \quad \text{for } \theta \leq 0.$$

Notice that the functions in both sides are normalized.

Suppose that Equation (9) holds for f and g in NBV. Then f is locally absolutely continuous on $(-\infty, 0)$, which implies that

$$\text{Var}(f, [s, t]) = \int_s^t |\dot{f}(\theta)| d\theta \quad \text{for } -\infty < s < t < 0.$$

Since f is of bounded variation on $(-\infty, 0]$, it follows that $|\dot{f}(\theta)|$ is integrable on $(-\infty, 0)$: that is, f is absolutely continuous on $(-\infty, 0)$. Furthermore, from Equation (9) we have that

$$(10) \quad \dot{f}(\theta) + \gamma f(\theta) = c - g(\theta) \quad \text{a.e. in } (-\infty, 0).$$

Since $f(0-) = h(0-) = \frac{c}{\gamma}$ from Equation (9), we obtain Relation (6); and the solution f of Equation (10) is

$$(11) \quad f(\theta) = \frac{c}{\gamma} + \int_{\theta}^0 e^{-\gamma(\theta-s)} g(s) ds \quad \text{for } \theta < 0.$$

Suppose that $\gamma > 0$. Since $f(-\infty)$ exists, we have that $\lim_{\theta \rightarrow -\infty} e^{\gamma\theta} [f(\theta) - c/\gamma] = 0$, which implies that

$$\int_{-\infty}^0 e^{\gamma s} g(s) ds = 0.$$

Hence f is rewritten as

$$f(\theta) = \frac{c}{\gamma} - \int_{-\infty}^{\theta} e^{-\gamma(\theta-s)} g(s) ds = \frac{c}{\gamma} - \int_{-\infty}^0 e^{\gamma t} g(t+\theta) dt,$$

from which it follows that

$$(12) \quad f(-\infty) = \frac{c}{\gamma} - \frac{g(-\infty)}{\gamma}.$$

Suppose $\gamma < 0$. Writing the integral in (11) as

$$\int_{\theta}^0 e^{-\gamma(\theta-s)} g(s) ds = \left[\int_{\theta}^N + \int_N^0 \right] e^{-\gamma(\theta-s)} g(s) ds \quad \theta < N < 0,$$

we have directly Relation (12), or

$$c = \gamma f(-\infty) + g(-\infty).$$

This relation and the definition of c imply Relation (5);

Equation (10) then becomes

$$\dot{f}(\theta) = -\gamma[f(\theta) - f(-\infty)] + g(\theta) - g(-\infty) \quad \text{a.e. in } (-\infty, 0).$$

This means that the equivalence class of \dot{f} in $L^1((-\infty, 0), E^*)$ contain a function which is in NBV and converges to 0 as θ

$\rightarrow -\infty$. Notice that such a function is unique for the equivalence class of \dot{f} since it is left continuous on $(-\infty, 0)$.

Conversely, suppose f has this property, and let $k(\theta)$ be the function, in $\text{NBV} \cap L^1((-\infty, 0), E^*)$, such that $\dot{f}(\theta) = k(\theta)$ a.e. in $(-\infty, 0)$ and that $k(-\infty) = 0$. Define $g(\theta)$ by $g(0) = 0$ and $g(\theta) = -\gamma[f(\theta) - f(0-)] - k(\theta)$ for $\theta < 0$. Then g is in NBV and $\dot{f}(\theta) = -\gamma[f(\theta) - f(0-)] - g(\theta)$ a.e. in $(-\infty, 0)$, which implies that

$$(13) \quad f(\theta) - f(0-) = -\gamma \int_0^\theta f(s) \, ds + \gamma \theta f(0-) - \int_0^\theta g(s) \, ds$$

for $\theta < 0$. Since $g(-\infty) = -\gamma[f(-\infty) - f(0-)] - k(-\infty) = -\gamma[f(-\infty) - f(0-)]$, we have that $\gamma f(0-) = \gamma f(-\infty) + g(-\infty)$. If a and b in E^* satisfy Relation (5), it then follows that $\gamma f(0-) = \gamma[f(-\infty) - a] + g(-\infty) - b$. Therefore relation (13) becomes Relation (9), as required.

Theorem 3 says that $j_\gamma^*(a, f)$ is in $\mathcal{D}(B^*)$ if and only if f is represented as

$$(14) \quad f(\theta) = d + \int_\theta^0 k(s) \, ds \quad \text{for } \theta < 0,$$

and for some d in E^* and for some k in $\text{NBV} \cap L^1((-\infty, 0), E^*)$ with $k(-\infty) = 0$. In this case the C_γ^* norm of $j_\gamma^*(a, f)$ is given by

$$|j_{\gamma}^*(a, f)| = |a| + \text{Var}(f, (-\infty, 0)) = |a| + |d| + \int_{-\infty}^0 |k(s)| \, ds.$$

Thus we have the following result, which is also valid in the case that $\gamma = 0$, [5].

Corollary 4.

$$C_{\gamma}^{\Theta} \simeq E^* \times E^* \times L^1((-\infty, 0), E^*).$$

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